Quantitative and Mathematical Methods

Euro-American Campus · Sciences Po · Reims

Session 5 · Review

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Level 1 Groups

Applications

1. Exponentials
2. Derivatives

... and we will run again through examples from previous sessions.

Exponentials

Definition

$y = b^x \qquad b > 0$ and $b \neq 1$

Natural exponential base e

$e = \lim_{x \to \infty} (1 + \frac{1}{x})^x \approx 2.718$

Logarithms

$y = \log_b x$ such that $b^y = x$

Natural logarithm: $ln(x)$

$y = ln(x)$ iff $e^y = x \qquad x > 0$

$ln(1) = c$ such as $e^c = 1$, and $ln(e) = c$ such as $e^c = e$

Rules

$\forall a, b \in \mathbb{R}+$ and $x, y \in \mathbb{R}$, the following rules apply:

- Equality: $b^x = b^y$ iff $x = y$
- Power: $(b^x)^y = b^{xy}$
- Product: $b^x b^y = b^{x+y}$
- Quotient: $\frac{b^x}{b^y} = b^{x-y}$
- Multiplication: $(ab)^x = a^x b^x$
- Division: $(\frac{a}{b})^x = \frac{a^x}{b^x}$

Session 3, Example 1: Population growth

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$$
\begin{aligned}
P(0) &= 6.1 \\
P(1) &= P(0) \times 1.014 \\
&= 6.1 \times (1.014)^1 \\
P(2) &= P(1) \times 1.014 = [6.1 \times (1.014)] \times (1.014) \\
&= 6.1 \times (1.014)^2 \\
P(3) &= 6.1 \times (1.014)^3 \\
&\vdots \\
P(t) &= 6.1 \times (1.014)^t
\end{aligned}
$$

Session 3, Example 2: Solving exponentials

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If a pricing function achieves $P(x) = 5^{x^2+2x}$, find all values of x such that $P(x) = 125$.

The equation is satisfied iff $5^{x^2+2x} = 5^3$.
Since $b^x = b^y$ iff $x = y$:

$$\begin{aligned} x^2 + 2x &= 3 \\ x^2 + 2x - 3 &= 0 \\ (x-1)(x+3) &= 0 \end{aligned}$$

$\rightarrow P(x) = 125$ iff $x = +1, x = -3$.

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$$\text{Solving } A : Q(0) = 15 \rightarrow Ae^{0} = A = 15$$

$$\text{Solving } k : Q(10) = 9 \rightarrow 9 = 15e^{-10k} \rightarrow \frac{3}{5} = e^{-10k}$$

$$\ln\frac{3}{5} = -10k \rightarrow k = -\frac{\ln 3/5}{10} \approx .05$$

Exponential function for population density: $Q(x) = 15e^{-.05x}$.

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Derivatives

Formula:

The derivative of $f(x)$ is the function $f'(x) = lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

Slope of a tangent:

The point $(c, f(c))$ at $m_{tan} = f'(c)$ is the slope of the tangent line to the curve $y = f(x)$ at c.

Significance of the sign:

- $f(x)$ is increasing at $x = c$ if $f'(c) > 0$
- $f(x)$ is decreasing at $x = c$ if $f'(c) < 0$

Rules

Constant rule:

$\frac{d}{dx}[c] = 0 \quad \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = 0$ if $f(x) = c$

Power rule:

$\frac{d}{dx}[x^n] = nx^{n-1}$

Constant multiple rule:

$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$

Sum rule:

$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$

More rules

Product rule:

$\frac{d}{dx}[f(x)g(x)] = f(x) \cdot \frac{d}{dx}[g(x)] + g(x) \cdot \frac{d}{dx}[f(x)]$

or equivalently: $(fg)' = fg' + gf'$

Quotient rule:

$(\frac{f}{g})' = \frac{gf' - fg'}{g^2}$

Second derivative

$f''(x) = \frac{d^2y}{dx^2}$ is the *second derivative* of $f'(x)$. The derivative of order n is denoted $f^{(n)}(x)$.

Chain rule

If $y = f(u)$ and $u = g(x)$, then $f(g(x)) = \frac{dy}{dx} = f'(g(x))g'(x)$

Session 4, Example 2: Population growth

Consider a population for which the growth function is $P(t) = t^2 + 20t + 8,000$ million people per year.
Find the growth rate at $t = 10$ and $t = 11$, and the actual change in population at $t = 11$.

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$$
\begin{aligned}
P'(t) &= 2t + 20 \\
P'(10) &= 2(10) + 20 = 40 \text{ people/year at } t = 10 \\
P'(11) &= 2(11) + 20 = 42 \text{ people/year at } t = 11 \\
P(10) &= 100 + 200 + 8,000 \\
P(11) &= 121 + 220 + 8,000 \\
P(11) - P(10) &= 41 \text{ people/year at } t = 11
\end{aligned}
$$

Application to population growth (continued)

Find the equations of the tangents at $t = 10$ and $t = 11$.

Application to population growth (continued)

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\begin{aligned}
P'(10) &= 2(10) + 20 = 40 \\
\text{at } t = 10, \quad y - f(10) &= 10^2 + 20(10) + 8,000 = 8,300 = 40(x - 10) \\
y &= 40x - 400 + 8,300 = 40x + 7,900 \\
P'(11) &= 2(11) + 20 = 42 \\
\text{at } t = 11, \quad y - f(11) &= 11^2 + 20(11) + 8,000 = 8,341 = 42(x - 11) \\
y &= 42x - 462 + 8,341 = 42x + 7,879
\end{aligned}
$$

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$$N'(10) = 2(10) + 5 = 25 \text{ billion dollars at } t = 10$$

What is the **relative** growth rate of GDP in that same year?

At $t = 10, N(10) = 100 + 50 + 101 = 251$ and $N'(10) = 25$.

The relative growth rate $\frac{Q'(x)}{Q(x)} = \frac{dQ/dx}{Q}$ is $\frac{25}{251} \approx 10\%$ per year in that period.

Session 4, Example 5: Worker productivity

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At $t = 3$, $R'(3) = Q''(3) = -6(3) + 12 = -6$ units/hour.

It might be a good idea to offer the worker a lunch break at that point.