Quantitative and Mathematical Methods
Euro-American Campus · Sciences Po · Reims

Session 4 · Derivatives

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Level 1 Groups

Applications

1. Profit function
2. Population growth rate
3. GDP growth rate
4. Wage increases
5. Worker productivity

Derivatives

The derivative $f'(x)$ expresses the **rate of change** in a function f. The aim is to measure the change in y for each change in x, denoted $\frac{dy}{dx}$ or $\frac{\delta_y}{\delta_x}$. We also might want to know the specific rate of change at a given value of x.

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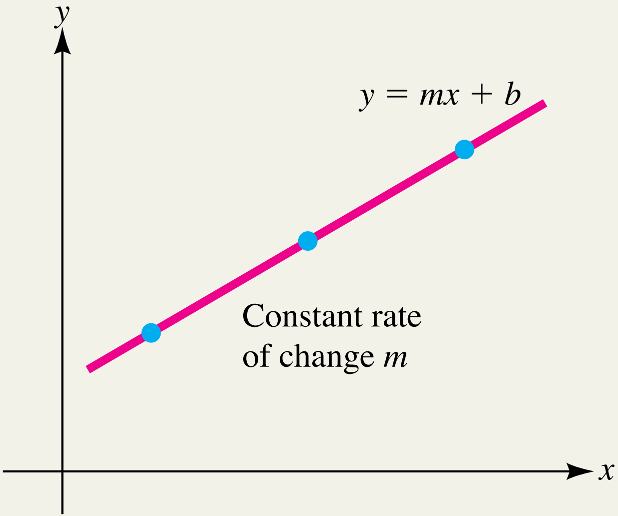
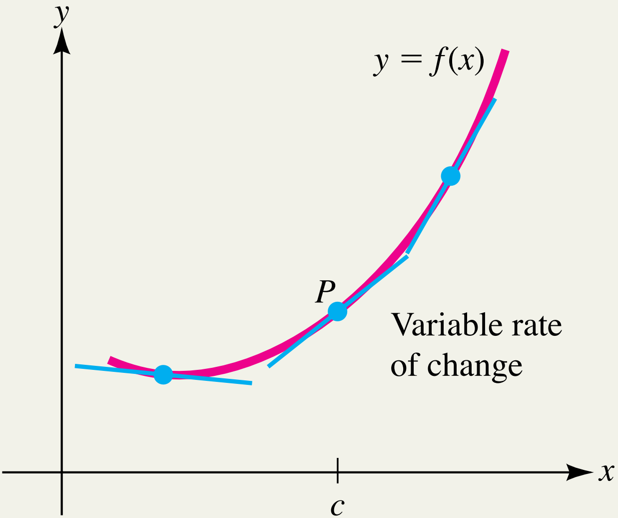
- If $f(x)$ is differentiable at $x = c$, then it is also continuous at $x = c$ (and 'near' it).

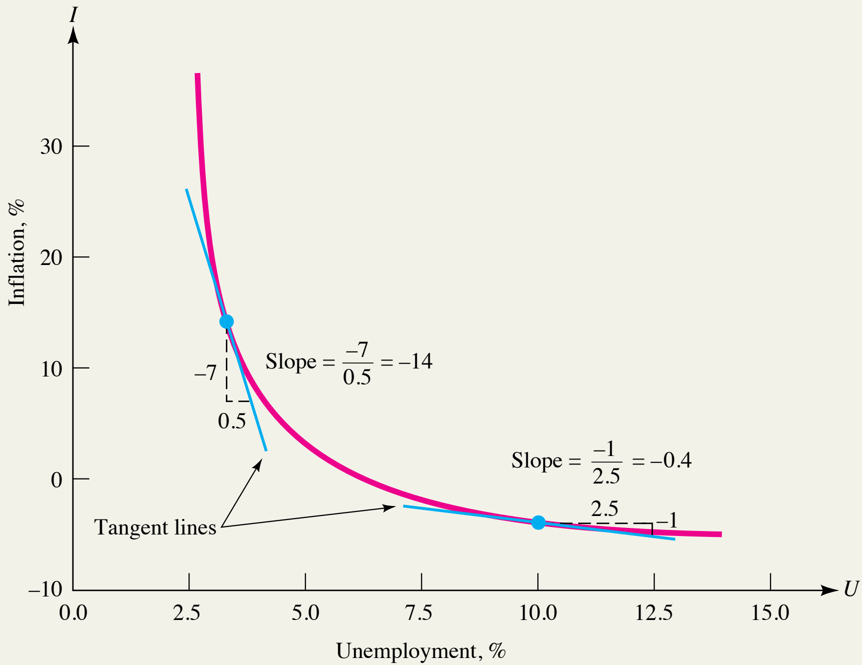
$f'(x)$ is solved analytically by a limit, and is graphically observable.



(a) A linear function $L(x) = mx + b$ changes at the constant rate m.



(b) If $f(x)$ is nonlinear, the rate of change at $x = c$ is given by the slope of the tangent line at $P(c, f(c))$.



I
30
20
10
0
–10
Inflation, %
–7
0.5
Slope = $\frac{-7}{0.5}$ = –14
Slope = $\frac{-1}{2.5}$ = –0.4
2.5
–1
Tangent lines
0.0
2.5
5.0
7.5
10.0
12.5
15.0
U
Unemployment, %

Expression of tangential change

Consider a risk that increases by $s(t) = 4t^2$ (%), with t in decades.

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What is the **average rate of change** during that decade?

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What is the **average rate of change** during that decade?

Compute the increase in risk v from $t = 2$ to $t = 2 + h$ for a time change of h, i.e. $\frac{\delta y}{\delta x}$:

$$\begin{aligned} v &= \frac{s(2+h) - s(2)}{(2+h) - 2} \\ &= \frac{4(2+h)^2 - 4(2)^2}{h} = \frac{4(4+4h+h^2) - 4(4)}{h} \\ &= \frac{16h + 4h^2}{h} = 16 + 4h \end{aligned}$$

Expression of tangential change

Analytical solving

Consider the smallest possible time change where $h \to 0$:

$\lim_{h \to 0} v = \lim_{h \to 0} 16 + 4h = 16$

The rate of change at $t = 2$ was, on average, 16% per decade.

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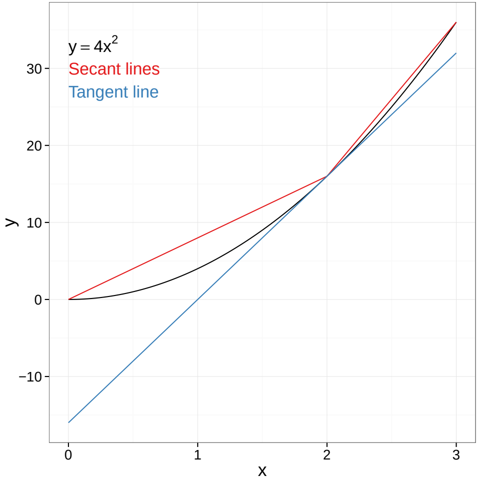
Graphical solving

Using the point and slope equation $y - y_0 = m(x - x_0)$, where m is the slope $\frac{y_2 - y_1}{x_2 - x_1} = 16$, $x_0 = t = 2$ and $y_0 = f(t) = 16$, then:

$$y - 16 = 16(x - 2) = 16x - 32$$

$$y = 16x - 16 = 16(x - 1)$$

Expression of tangential change



Secant lines approach the rate of change at $t = 2 \pm h$.

As $h \to 0$, one secant tends towards the tangent, $f'(x)$.

Final definitions

Formula:

The derivative of $f(x)$ is the function $f'(x) = lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$.

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Significance of the sign:

- $f(x)$ is increasing at $x = c$ if $f'(c) > 0$
- $f(x)$ is decreasing at $x = c$ if $f'(c) < 0$

Economic profitability

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$$
\begin{aligned}
P'(x) &= \lim_{h\to 0} \frac{P(x+h) - P(x)}{h} \\
&= \lim_{h\to 0} \frac{[-4(x+h)^2 + 60(x+h) - 12] - (-4x^2 + 60x - 12)}{h} \\
&= \lim_{h\to 0} \frac{-4h^2 - 8hx + 60h}{h} \\
&= \lim_{h\to 0} -4h - 8x + 60 = 60 - 8x
\end{aligned}
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The profit rate at $x = 10$ is $P'(10) = 60 - 8(10) = -20$ dollars per unit of x.

Economic profitability (continued)

Find the equation of the tangent at $x = 10$.

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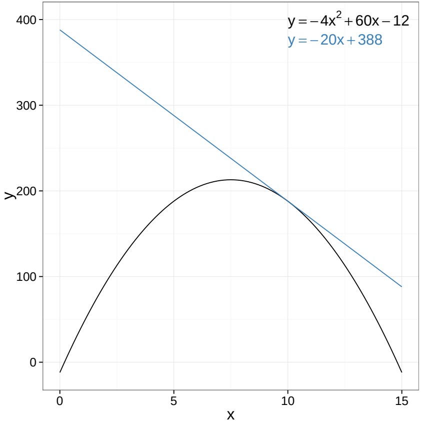

Function:

$f(x) = -4x^2 + 60x - 12$

Derivative:

$f'(x) = 60 - 8x$

Slope of tangent at $x = 10$:

$60 - 8(10) = -20$

Economic profitability (continued)

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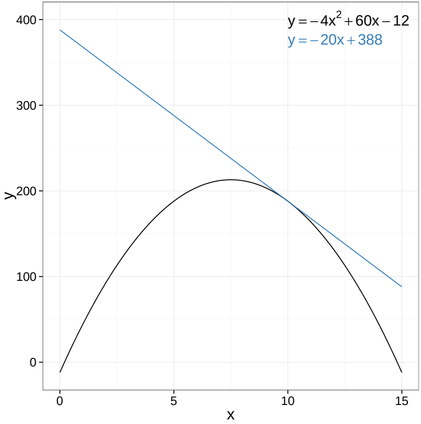

Function:

$$f(x) = -4x^2 + 60x - 12$$

Derivative:

$$f'(x) = 60 - 8x$$

Slope of tangent at $x = 10$:

$$60 - 8(10) = -20$$

$$\begin{aligned} f(10) &= -4(10^2) + 60(10) - 12 \\ &= 188 \end{aligned}$$

$$y - 188 = -20(x - 10)$$

$$y = -20x + 388$$

Rules

Constant rule:

$\frac{d}{dx}[c] = 0 \quad \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = 0$ if $f(x) = c$

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Sum rule:

$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$

Application to population growth

Consider a population for which the growth function is
$P(t) = t^2 + 20t + 8,000$ million people per year.
Find the growth rate at $t = 10$ and $t = 11$, and the actual change in population at $t = 11$.

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Find the growth rate at $t = 10$ and $t = 11$, and the actual change in population at $t = 11$.

$$
\begin{aligned}
P'(t) &= 2t + 20 \\
P'(10) &= 2(10) + 20 = 40 \text{ people/year at } t = 10 \\
P'(11) &= 2(11) + 20 = 42 \text{ people/year at } t = 11 \\
P(10) &= 100 + 200 + 8,000 \\
P(11) &= 121 + 220 + 8,000 \\
P(11) - P(10) &= 41 \text{ people/year at } t = 11
\end{aligned}
$$

Application to population growth (continued)

Find the equations of the tangents at $t = 10$ and $t = 11$.

Application to population growth (continued)

Find the equations of the tangents at $t = 10$ and $t = 11$. Using the point-slope equation $y - y_0 = m(x - x_0)$:

$$P'(10) = 2(10) + 20 = 40 \quad \rightarrow \text{slope} m = 40$$

$$\text{at } t = 10, \quad y - f(10) = y - 10^2 + 20(10) + 8,000 = 8,300$$

$$= 40(x - 10)$$

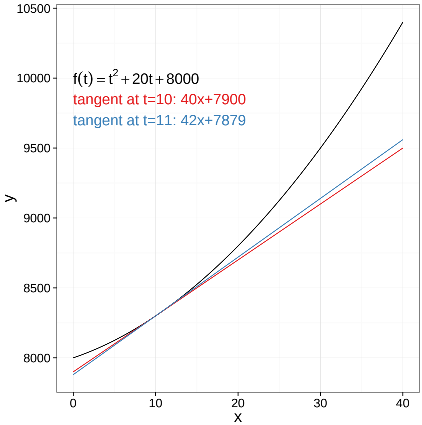
$$y = 40x - 400 + 8,300 = 40x + 7,900$$

$$P'(11) = 2(11) + 20 = 42$$

$$\text{at } t = 11, \quad y - f(11) = y - 11^2 + 20(11) + 8,000 = 8,341 = 42(x - 1$$

$$y = 42x - 462 + 8,341 = 42x + 7,879$$

Application to population growth (graphical check)



Relative rates

Relative rate of change:

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e.g.

- if the slope of the tangent at x is $c = -20$, then the function is decreasing, since $f'(c) < 0$
- if the actual value of x is $2,000$, then the function is decreasing at a rate of $\frac{20}{2{,}000} = .01$, or 1%

GDP growth rate

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$$N'(10) = 2(10) + 5 = 25 \text{ billion dollars at } t = 10$$

What is the **relative** growth rate of GDP in that same year?

At $t = 10, N(10) = 100 + 50 + 101 = 251$ and $N'(10) = 25$.
The relative growth rate is $\frac{25}{251} \approx 10\%$ per year in that period.

Wage increases

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$$\text{at } t = 1, \quad \frac{100f'(t)}{f(t)} = \frac{200}{50} = 4\%$$

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The percentage rate of change decreases with time as $t \to +\infty$.

More rules

Product rule:

$$\frac{d}{dx}[f(x)g(x)] = f(x) \cdot \frac{d}{dx}[g(x)] + g(x) \cdot \frac{d}{dx}[f(x)]$$

or equivalently: $(fg)' = fg' + gf'$

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$(\frac{f}{g})' = \frac{gf' - fg'}{g^2}$

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Second derivative

$f''(x) = \frac{d^2y}{dx^2}$ is the *second derivative* of $f'(x)$. The derivative of order n is denoted $f^{(n)}(x)$.

Chain rule

If $y = f(u)$ and $u = g(x)$, then $f(g(x)) = \frac{dy}{dx} = f'(g(x))g'(x)$ (next week)

Product rule

Demonstrate that the product rule for $f(x) = x^2$ and $g(x) = x^3$ does not work like the multiple and sum rules.

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$$f(x)g(x) = x^2x^3 = x^5$$
$$[f(x)g(x)]' = 5x^4$$

As proof that the product rule does not work like like the multiple and sum rules:

$$f'(x) = 2x$$
$$g'(x) = 3x^2$$
$$f'(x)g'(x) = (2x)(3x^2) = 6x^3$$

Product rule

Example 1: Apply the product rule to $f(x) = x^2$ and $g(x) = x^3$.

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$$\begin{aligned}(x^2x^3)' &= x^2(x^3)' + x^3(x^2)' \\ &= x^2(3x^2) + x^3(2x) = 3x^4 + 2x^4 = 5x^4\end{aligned}$$

Product rule

Example 2: Differentiate $P(x) = (x - 1)(3x - 2)$.

Product rule

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$$P(x) = 3x^2 - 5x + 2$$
$$P'(x) = 6x - 5$$

Using the product rule:

$$P'(x) = (x-1)(3x-2)' + (3x-2)(x-1)'$$
$$= (x-1)(3) + (3x-2)(1) = 6x - 5$$

Product rule

Example 3.1: differentiate $y(x) = (2x + 1)(2x^2 - x - 1)$.

Product rule

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$$\begin{aligned} y'(x) &= (2x+1)[2x^2-x-1]' + (2x^2-x-1)[2x+1]' \\ &= (2x+1)(4x-1) + (2x^2-x-1)(2) \end{aligned}$$

Product rule

$$y(x) = (2x + 1)(2x^2 - x - 1)$$
$$y'(x) = (2x + 1)(4x - 1) + (2x^2 - x - 1)(2)$$

Example 3.2: find the equation for the tangent line at $y = 1$.

Product rule

$$y(x) = (2x+1)(2x^2 - x - 1)$$
$$y'(x) = (2x+1)(4x-1) + (2x^2 - x - 1)(2)$$

Example 3.2: find the equation for the tangent line at $y = 1$.

$$y(1) = (2(1)+1)(2(1)^2 - (1) - 1) = 0 \quad \text{tangency at point (1,0)}$$
$$y'(1) = (2(1)+1)(4(1)-1) + (2(1)^2 - (1) - 1)(2) = 9$$
$$y - 0 = 9(x-1) = 9x - 9 \quad \text{using the point-slope equation}$$

Product rule

$$y'(x) = (2x + 1)(4x - 1) + (2x^2 - x - 1)(2)$$

Example 3.3: identify horizontal tangents (null growth) by solving $y' = 0$.

Product rule

$$y'(x) = (2x+1)(4x-1) + (2x^2 - x - 1)(2)$$

Example 3.3: identify horizontal tangents (null growth) by solving $y' = 0$.

$$\begin{aligned} y'(x) &= 12x^2 - 3 \quad \text{by polynomial expansion} \\ 12x^2 - 3 &= 0 \\ x^2 &= \frac{3}{12} = \frac{1}{4} \quad \text{therefore } x = \frac{1}{2} \text{ and } x = -\frac{1}{2} \end{aligned}$$

Second derivatives

Calculate $f''(x)$ for:

$f(x) = 5x^4 - 3x^2 - 3x + 7$

$g(x) = x^2(3x + 1)$

Second derivatives

Calculate $f''(x)$ for:

$f(x) = 5x^4 - 3x^2 - 3x + 7$

$g(x) = x^2(3x + 1)$

$$f'(x) = 20x^3 - 6x - 3$$

$$f''(x) = 60x^2 - 6$$

$$g'(x) = x^2(3) + (3x + 1)(2x) = 9x^2 + 2x$$

$$g''(x) = 18x + 2$$

Worker productivity

If a worker has a unit productivity function of $Q(t) = -t^3 + 6t^2 + 24t$ at 8am, what is his unit productivity at 11am, and at what rate is it changing by that time?

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Rate of production: $R(t) = Q'(t) = -3t^2 + 12t + 24$ of $Q(t)$.

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At $t = 3$, $R(3) = Q'(3) = -3(3)^2 + 12(3) + 24 = 33$ units/hour.

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At $t = 3$, $R(3) = Q'(3) = -3(3)^2 + 12(3) + 24 = 33$ units/hour.
The change in the rate of production is $R'(t) = Q''(t) = -6t + 12$.

Worker productivity

If a worker has a unit productivity function of $Q(t) = -t^3 + 6t^2 + 24t$ at 8am, what is his unit productivity at 11am, and at what rate is it changing by that time?

Rate of production: $R(t) = Q'(t) = -3t^2 + 12t + 24$ of $Q(t)$.
At $t = 3$, $R(3) = Q'(3) = -3(3)^2 + 12(3) + 24 = 33$ units/hour.
The change in the rate of production is $R'(t) = Q''(t) = -6t + 12$.
At $t = 3$, $R'(3) = Q''(3) = -6(3) + 12 = -6$ units/hour.

It might be a good idea to offer the worker a lunch break at that point.

Next week

- Differentiation using the chain rule
- Differentiation of exponentials and logarithms
- Application of marginal effects and differentials

Make sure that you are up-to-date on your handbook readings: next week is the last session before essential calculus exam.