Quantitative and Mathematical Methods

Euro-American Campus · Sciences Po · Reims

Session 3 · Exponentials

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Level 1 Groups

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Today: powers, exponentials, logarithms.

Examples:

Example 1: Population growth

Consider an initial population of 6.1 billion people at $P(0) = 2000$, and a constant annual growth rate of 1.4%. Find $P(t)$.

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Consider an initial population of 6.1 billion people at $P(0) = 2000$, and a constant annual growth rate of 1.4%. Find $P(t)$.

$$
\begin{aligned}
P(0) &= 6.1 \\
P(1) &= P(0) \times 1.014 \\
&= 6.1 \times (1.014)^1 \\
P(2) &= P(1) \times 1.014 = [6.1 \times (1.014)] \times (1.014) \\
&= 6.1 \times (1.014)^2 \\
P(3) &= 6.1 \times (1.014)^3 \\
&\vdots \\
P(t) &= 6.1 \times (1.014)^t
\end{aligned}
$$

Power functions

Integer powers

If $k \in \mathbb{Z}^{+}, p(x, k) = x^{k}$ k is the *exponent*.

e.g. $2^3 = 2 \times 2 \times 2 = 8$

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Fractional powers

If $n, m \in \mathbb{Z}^+, x^{\frac{n}{m}} = (\sqrt[m]{x})^n$ $\quad \sqrt[m]{x}$ is the positive m-th root of x.

$e.g.\quad 4^{\frac{1}{2}} = (\sqrt[2]{4})^1 = \sqrt{4} = 2$

$e.g.\quad 4^{\frac{3}{2}} = (\sqrt[2]{4})^3 = 2^3 = 8$

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Negative and null powers

At negative powers, $x^{-k} = \frac{1}{x^n}$

At zero power, $x^0 = 1$.

Exponential functions

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e.g. $y = 2^x$

$2^{-10} = .001 \quad 2^{-3} = .125 \quad 2^{-1} = .5 \quad 2^0 = 1 \quad 2^1 = 2 \quad 2^3 = 8$

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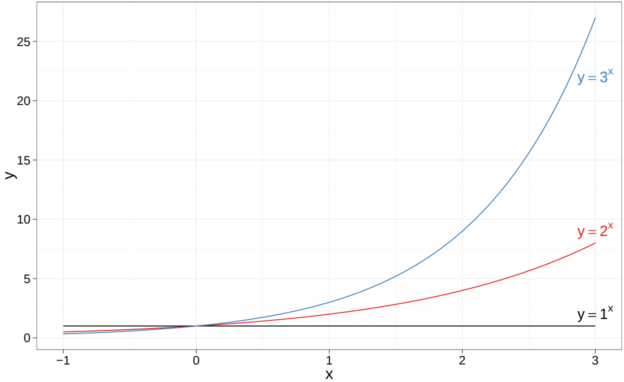
e.g. $y = (\frac{1}{2})^x$

$(\frac{1}{2})^{-10} = 1,024 \quad (\frac{1}{2})^{-3} = 8 \quad (\frac{1}{2})^0 = 1 \quad \frac{1}{2}^2 = .25, \frac{1}{2}^3 = .125$

$\rightarrow f(x) = (\frac{1}{2})^x$ is always decreasing

$y = b^x$ with $b > 1$

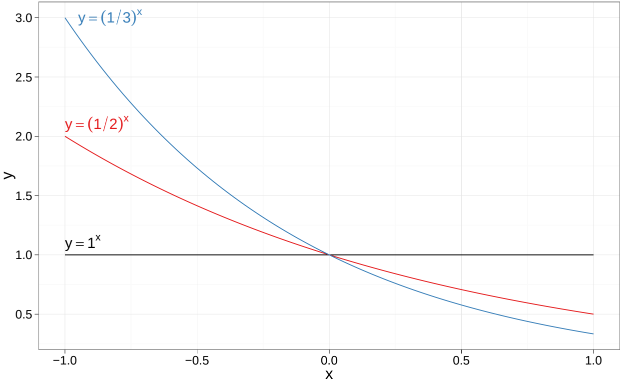
Exponential growth: $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to +\infty} f(x) = +\infty$



Note that x intercept $= \{\varnothing\}$, y intercept $= 1$.

$y = b^x$ with $0 < b < 1$

Exponential decay: $\lim_{x \to -\infty} f(x) = +\infty$ and $\lim_{x \to +\infty} f(x) = 0$



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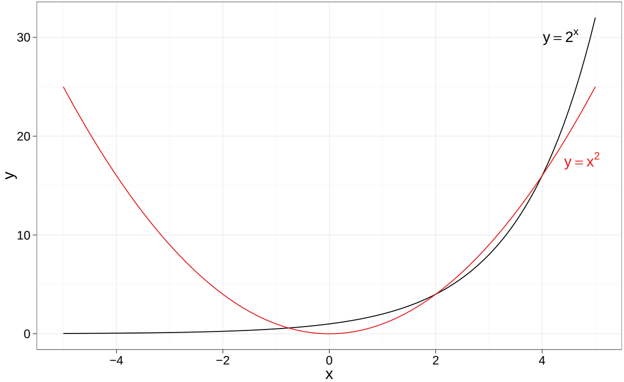
Warning powers $\neq$ exponentials

$f(x) = b^x$ is an exponential function of base b

$g(x) = x^b$ is power function of exponent b



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- Product: $b^x b^y = b^{x+y}$
- Quotient: $\frac{b^x}{b^y} = b^{x-y}$
  e.g. $\frac{2^3}{2^5} = 2^{3-5} = 2^{-2} = \frac{1}{4}$

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- Multiplication: $(ab)^x = a^x b^x$
- Division: $(\frac{a}{b})^x = \frac{a^x}{b^x}$
  e.g. $(\frac{2}{5})^3 = \frac{2^3}{5^3} = \frac{8}{125}$

Example 2: Solving exponentials

If $f(x) = 5^{x^2+2x}$, find all values of x such that $f(x) = 125$.

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The equation is satisfied iff $5^{x^2+2x} = 5^3$.
Since $b^x = b^y$ iff $x = y$:

$$\begin{aligned} x^2 + 2x &= 3 \\ x^2 + 2x - 3 &= 0 \\ (x-1)(x+3) &= 0 \end{aligned}$$

$\rightarrow f(x) = 125$ iff $x = +1, x = -3$.

Natural exponential base e

$f(x) = (1 + \frac{1}{x})^x$

$$\begin{aligned} f(10) &= 1.1^{10} && \approx 2.59 \\ f(100) &= 1.01^{100} && \approx 2.70 \\ f(1000) &= 1.001^{1000} && \approx 2.71 \end{aligned}$$

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Example 3a: Compound interest

Assume P is a sum invested at an annual interest rate of r.

The balance after compounding once is $B = P + P \cdot r = P(1 + r)$.

If compounding occurs k times per year, then the interest rate is $\frac{r}{k}$.

Find the annual balance function $B(t)$.

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$$P_1 = P_0 + P_0(\frac{r}{k}) = P_0(1 + \frac{r}{k})$$

$$P_2 = P_1 + P_1(\frac{r}{k}) = P_1(1 + \frac{r}{k})$$

$$= [P_0(1 + \frac{r}{k})](1 + \frac{r}{k}) = P_0(1 + \frac{r}{k})^2$$

$$\rightarrow P(t) = P_0(1 + \frac{r}{k})^{kt}$$

Example 3b: Continuous compound interest

$\rightarrow B(t) = P(1 + \frac{r}{k})^{kt}$

What happens if k goes to infinity?

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What happens if k goes to infinity?

Let $k = nr$.

$$
\begin{aligned}
P(t) &= P_0(1 + \frac{r}{k})^{kt} \\
&= P_0(1 + \frac{r}{nr})^{nrt} = P_0[(1 + \frac{1}{n})^n]^{rt} \\
\lim_{t \to \infty} P(t) &= P_0[\lim_{n \to \infty} (1 + \frac{1}{n})^n]^{rt} \\
\rightarrow B(t) &= Pe^{rt}
\end{aligned}
$$

Independently of k, the balance will never exceed Pe^{rt}.

Example 3c: Present value

$\rightarrow B(t) = P(1 + \frac{r}{k})^{kt}$

Assume that we know the *future* value $B(t)$ that we want to accumulate over time. How much do we need to invest at $t = 0$ to obtain $B(t)$?

Find P, i.e. the *present* value of B to be received in t years.

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$$B(t) = P(1+\frac{r}{k})^{kt}$$

$$B(t)(1+\frac{r}{k})^{-kt} = P(1+\frac{r}{k})^{kt}(1+\frac{r}{k})^{-kt}$$

Since $b^x b^y = b^{x+y}, P = B(t)(1+\frac{r}{k})^{-kt}$

If compounding is continuous, $B = Pe^{rt}$ and therefore $P = Be^{-rt}$.

Logarithmic functions

Exponential to the base b

$y = b^x \quad b > 0, b \neq 1$

e.g. $2^3 = 8$

e.g. $10^4 = 10,000$

e.g. $5^{-3} = \frac{1}{5^3} = \frac{1}{125}$

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Logarithm of x to the base b

$y = \log_b x$ such that $b^y = x$

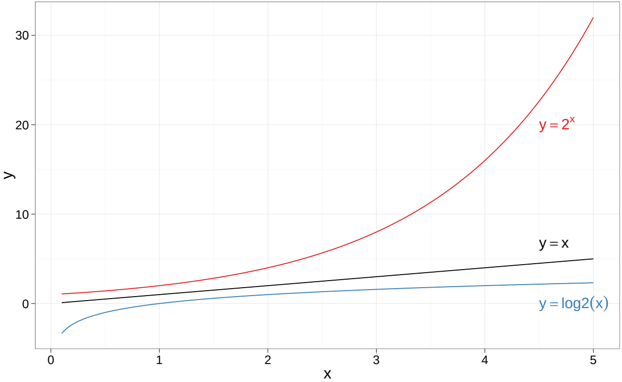
e.g. $\log_2 8 = 3$

e.g. $\log_{10} 10,000 = 4$

e.g. $\log_5 \frac{1}{125} = -3$

$y = b^x$ with $b > 1$

Logarithms reverse the process of exponentiation, which allows to express products and quotients as sums and differences:



Rules for logarithms

Let b be any logarithmic base $(b > 0, b \neq 1)$.

Then $\log_b 1 = 0$ since $b^0 = 1$ and $\log_b b = 1$ since $b^1 = 0$.

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- Equality: $\log_b x = \log_b^y$ iff $x = y$
- Power: $\log_b x^k = k \log_b x \quad \forall k \in \mathbb{R}$

  e.g. $\log_5 8 = \log_5 2^3 = 3 \log_5 2$

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- Product: $\log_b xy = \log_b x + \log_b y$
- Quotient: $\log_b \frac{x}{y} = \log_b x - \log_b y$
  e.g. $\log_2 \frac{7}{3} = \log_2 7 - \log_2 3$

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- Inversion: $\log_b b^u = u$

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$x^3 = 27 \rightarrow x = 27^{\frac{1}{3}} = \sqrt[3]{27} = 3.$

$\log_{64} 16 = x$

$$64^x = 16$$
$$(2^6)^x = 2^4$$
$$6x = 4 \text{ since } b^x = b^y \text{ implies } x = y \rightarrow x = \frac{2}{3}$$

Example 4b: Rewriting logarithms

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Rewrite each of the following expressions in $\log_5 2$ and $\log_5 3$:

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$\log_5 64 = \log 52^6 = 6 \log_5 2$.

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$\log_5 36$

$$\begin{aligned} \log_5 36 &= \log_5 2^2 \cdot 3^2 \\ &= \log_5 2^2 + \log_5 3^2 \\ &= 2 \log_5 2 + 2 \log_5 3 \end{aligned}$$

Example 4c: Proving logarithms

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If $\log_b x = \log_b y$, then $x = y$.

Let $X = \log_b x$ and $Y = \log_b y$.
Then, by definition, $b^X = x$ and $b^Y = y$.
Therefore, if $\log_b x = \log_b y$, then $x = y$, so that:

$$b^X = b^Y$$

$$x = y$$

Example 4c: Proving logarithms (continued)

Prove the product rule

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Example 4c: Proving logarithms (continued)

Prove the product rule

If $\log_b xy = \log_b x + \log_b y$.

$$\begin{aligned}\log_b x + \log_b y &= X + Y \\ &= \log_b b^{X+Y} \\ &= \log_b b^X b^Y \\ &= \log_b xy\end{aligned}$$

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Remarkable values

$ln(1) = c$ such as $e^c = 1 \rightarrow$ since $e^0 = 1$, $ln(1) = 0$.
$ln(e) = c$ such as $e^c = e \rightarrow$ since $e^1 = e$, $ln(e) = 1$.

Solving capabilities

e.g. Let a super-quick exponential $e^{20x} = 3$.

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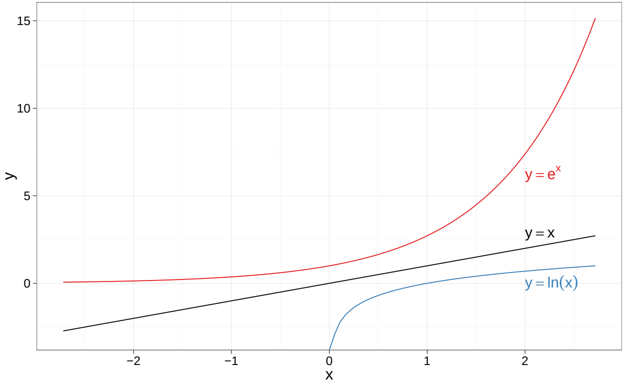
$y = \ln(x)$ with $x > 0$

Following the (linear) increase in $\ln(x)$ to track the (exponential) increase of e^x allows to model a wide range of nonlinear processes.



Urban density

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$$\text{Solving } A : Q(0) = 15 \rightarrow Ae^{0} = A = 15$$

$$\text{Solving } k : Q(10) = 9 \rightarrow 9 = 15e^{-10k} \rightarrow \frac{3}{5} = e^{-10k}$$

$$\ln\frac{3}{5} = -10k \rightarrow k = -\frac{\ln 3/5}{10} \approx .05$$

Exponential function for population density: $Q(x) = 15e^{-.05x}$.

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