Review exercises

Exponentials

Definition

$y = b^x \quad b > 0$ and $b \neq 1$

Natural exponential base e

$e = \lim_{x\to\infty} (1 + \frac{1}{x})^x \approx 2.718$

Logarithms

$y = \log_b x$ such that $b^y = x$

Natural logarithm: $ln(x)$

$y = ln(x)$ iff $e^y = x \quad x > 0$

$ln(1) = c$ such as $e^c = 1$, and $ln(e) = c$ such as $e^c = e$

Rules

$\forall a, b \in \mathbb{R}+$ and $x, y \in \mathbb{R}$, the following rules apply:

- Equality: $b^x = b^y$ iff $x = y$
- Power: $(b^x)^y = b^{xy}$
- Product: $b^x b^y = b^{x+y}$
- Quotient: $\frac{b^x}{b^y} = b^{x-y}$
- Multiplication: $(ab)^x = a^x b^x$
- Division: $(\frac{a}{b})^x = \frac{a^x}{b^x}$

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$$
\begin{aligned}
P(0) &= 6.1 \\
P(1) &= P(0) \times 1.014 \\
&= 6.1 \times (1.014)^1 \\
P(2) &= P(1) \times 1.014 = [6.1 \times (1.014)] \times (1.014) \\
&= 6.1 \times (1.014)^2 \\
P(3) &= 6.1 \times (1.014)^3 \\
&\vdots \\
P(t) &= 6.1 \times (1.014)^t
\end{aligned}
$$

Session 3, Example 5: Urban density

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$$\text{Solving } A: Q(0) = 24 \rightarrow Ae^0 = A = 24$$

$$\text{Solving } k: Q(5) = 6 \rightarrow 6 = 24e^{-5k} \rightarrow \frac{1}{4} = e^{-5k}$$

$$\ln\frac{1}{4} = -5k \rightarrow k = -\frac{\ln 1/4}{5} \approx .02$$

Exponential function for population density: $Q(x) = 24e^{-.02x}$.

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Derivatives

Formula:

The derivative of $f(x)$ is the function $f'(x) = lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

Slope of a tangent:

The point $(c, f(c))$ at $m_{tan} = f'(c)$ is the slope of the tangent line to the curve $y = f(x)$ at c.

Significance of the sign:

- $f(x)$ is increasing at $x = c$ if $f'(c) > 0$
- $f(x)$ is decreasing at $x = c$ if $f'(c) < 0$

Rules

Constant rule:

$\frac{d}{dx}[c] = 0 \quad \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = 0$ if $f(x) = c$

Power rule:

$\frac{d}{dx}[x^n] = nx^{n-1}$

Constant multiple rule:

$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$

Sum rule:

$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$

More rules

Product rule:

$\frac{d}{dx}[f(x)g(x)] = f(x) \cdot \frac{d}{dx}[g(x)] + g(x) \cdot \frac{d}{dx}[f(x)]$

Quotient rule:

$(\frac{f}{g})' = \frac{gf' - fg'}{g^2}$

Second derivative

The derivative of order n is denoted $f^{(n)}(x)$.

$f''(x) = \frac{d^2y}{dx^2}$ is the *second derivative* of $f'(x)$.

Chain rule

If $y = f(u)$ and $u = g(x)$, then $f(g(x)) = \frac{dy}{dx} = f'(g(x))g'(x)$

Session 4, Example 2: Population growth

Consider a population for which the growth function is
$P(t) = t^3 + t^2 + 12t + 1,000$ million people per year.
Find the growth rate at $t = 3$ and $t = 4$, and the actual change in population at $t = 4$.

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$$
\begin{aligned}
P'(t) &= 3t^2 + 2t + 12 \\
P'(3) &= 3(3)^2 + 2(3) + 12 = 45 \text{ people/year at } t = 3 \\
P'(4) &= 3(4)^2 + 2(4) + 12 = 68 \text{ people/year at } t = 4 \\
P(3) &= 27 + 9 + 12(3) = 1,072 \\
P(4) &= 64 + 16 + 12(4) = 1,128 \\
P(4) - P(3) &= 56 \text{ people/year at } t = 4
\end{aligned}
$$

Application to population growth (continued)

Find the equations of the tangents at $t = 3$ and $t = 4$.

Application to population growth (continued)

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$$\begin{aligned} P'(3) &= 45 \\ \text{at } t = 3, \quad y - f(3) &= 45(x - 3) \\ y &= 45x - 135 + f(3) = 45x + 937 \\ P'(4) &= 68 \\ \text{at } t = 4, \quad y - f(4) &= 68(x - 4) \\ y &= 68x - 272 + f(4) = 68x + 856 \end{aligned}$$

Session 4, Example 3: GDP growth rate

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What is the *relative* growth rate of GDP in that same year?

At $t = 10, N(10) = 100 + 50 + 101 = 251$ and $N'(10) = 25$.

The relative growth rate $\frac{Q'(x)}{Q(x)} = \frac{dQ/dx}{Q}$ is $\frac{25}{251} \approx 10\%$ per year in that period.

Session 4, Example 5: Worker productivity

If a worker has a unit productivity function of $Q(t) = -t^3 + 6t^2 + 24t$ at 8am, what is his unit productivity at 11am, and at what rate is it changing by that time?

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Rate of production: $R(t) = Q'(t) = -3t^2 + 12t + 24$ of $Q(t)$.

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At $t = 3$, $R'(3) = Q''(3) = -6(3) + 12 = -6$ units/hour

It might be a good idea to offer the worker a lunch break at that point.

Discrete probability

Probability of a random variable x

$P(x) = \frac{n(x)}{n(S)} \quad 0 \leq P \leq 1$

Expected value $E(x)$, or mean μ

The mean measures the average numerical outcome of x.
$E(x) = \sum_{i=1}^{n} x_i P(x_i)$.

Variance $V(x)$ or σ^2, and standard deviation σ

Variability measures the squared sum of deviations from the mean.
$V(x) = \sum_{i=1}^{n} (x_i - \mu)^2 P(x_i) \quad \sigma_x = \sqrt{V(x)}$

Continuous variables

Probability density function of x

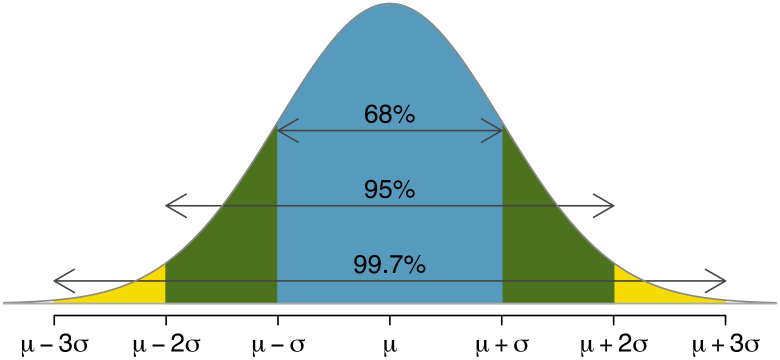
$P(a \leq x \leq b) = \int_a^b f(x)\,\mathrm{d}x \quad \int_{min}^{max} f(x)\,\mathrm{d}x = 1$

Standard normal distribution $\mathcal{N}(0, 1)$

- approx. 68% of values at $\mu \pm 1\sigma$
- approx. 95% of values at $\mu \pm 2\sigma (Z = 1.96)$
- approx. 99% of values at $\mu \pm 3\sigma (Z = 2.58)$

Standardized score: $Z = \frac{x-\mu}{\sigma}$ Standard error: $SEM = \frac{\sigma}{\sqrt{N}}$

Standard normal distribution



Source: Diez *et al.* 2011

Bernoulli trials: successes over a dichotomous outcome

Bernoulli variable

p is the proportion of successes in $S = \{0, 1\}$

$\mu = p \quad \sigma = \sqrt{p(1-p)}$

Probability of a single success after n trials

$(1-p)^{n-1}p$ (exponentially decreasing geometric distribution)

Mean, variance and standard deviation

$\mu = \frac{1}{p} \quad \sigma^2 = \frac{1-p}{p^2} \quad \sigma = \sqrt{\frac{1-p}{p}}$

Binomial distribution: n independent Bernoulli trials

Probability of a single success k out of n trials

$P(x = 1) = p^k(1 - p)^{n-k}$

Probability of k successes

$P(x = k) = \binom{n}{k} p^k(1 - p)^{n-k} \quad \binom{n}{r} = \frac{n!}{r!\,(n-r)!}$

Mean, variance and standard deviation

$\mu = np \quad \sigma^2 = np(1 - p) \quad \sigma = \sqrt{np(1 - p)}$

Normal approximation

If np and $n(1 - p)$ are both at least 10, the approximate normal distribution has parameters corresponding to the mean and standard deviation of the binomial distribution.

Exercise with discrete probabilities

Exercise 1: Trader gains

A trader wins \$500 on 80% of his transactions and loses \$1,000 on 10% of them. (a) How much does he gain on average? (b) At what probability is his expected gain null?

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$$\begin{aligned} E(x) &= \sum xp(x) \\ &= (.8)(500) + (.2)(-1000) \\ &= 200 \end{aligned}$$

(a) The trader expects to gain \$200 per transaction on average.
(b) $E(x) = 0$ when $500(p) = 1000(1-p)$, i.e. when $p = \frac{10}{15} \approx .6$.

Exercises with discrete probabilities

Exercise 2: Sexual transmission

If 40% of a population is contaminated with a sexual disease and sex acts occur at random, what percentage of sex acts poses a contamination risk?

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$$P(x_1 = 0, x_2 = 0) = (.6)(.6) = .36$$
$$P(x_1 = 1, x_2 = 1) = (.4)(.4) = .16$$
$$P(x_1 = 0, x_2 = 1) = (.6)(.4) = .24$$
$$P(x_1 = 1, x_2 = 0) = (.4)(.6) = .24$$

If sex acts within the population are random, 48% of sex acts expose one of the partners (x_1, x_2) to contamination.

Exercise with Bernoulli trials

Exercise 3: Equation review

Show that a Bernoulli variable has a mean of $\mu = p$ and a standard deviation of $\sigma = \sqrt{p(1-p)}$.

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Show that a Bernoulli variable has a mean of $\mu = p$ and a standard deviation of $\sigma = \sqrt{p(1-p)}$.

$$\begin{aligned}\mu &= E(x) = 0 \cdot P(x=0) + 1 \cdot P(x=1) \\ &= 0(1-p) + p = p \\ \sigma^2 &= P(x=0)(0-p)^2 + P(x=1)(1-p)^2 \\ &= (1-p)p^2 + p(1-p)^2 = p(1-p)\end{aligned}$$

Bernoulli trials are 'one dimension' of binomial distributions, which are equivalent to running several independent Bernoulli trials; for a binomial distribution, $\mu = np$ and $\sigma = \sqrt{np(1-p)}$.

Exercises with binomial probabilities

Exercise 4: Death rate

If 10% of smokers die within fifteen years, calculate the probabilities that out of a random sample of 20 smokers, (a) none will die within 15 years. (b) at least one will die within 15 years.

Exercises with binomial probabilities

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$$P(k) = \binom{20}{k}(.10)^k(1 - .10)^{20-k}$$

$$P(0) = \binom{20}{0} \cdot .1^0 \cdot (.9)^2 0 = \approx .12$$

$$P(x > 0) = 1 - P(0) \approx .88$$

Exercises with binomial probabilities

Exercise 5: Probability sampling

Assume a sample of 50 people randomly selected from a population of which a third opposes homogamy. (a) Calculate the mean and standard deviation of opposition to gay marriage in the sample. (b) How likely is it that the sample contains a maximum of three opponents to gay marriage?

Exercises with binomial probabilities

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$$\mu = np = (50)(1/3) \approx 16$$

$$\sigma^2 = np(1-p) = (50)(1/3)(2/3) \approx 11 \quad \sigma \approx \sqrt{11} \approx 3.3$$

(a) The sample will contain 16 opponents on average.

(b) It seems highly unlikely: given μ and σ, we would rather expect p to be within $[\mu - 2\sigma, \mu + 2\sigma]$, somewhere around $[10, 22]$.

Exercises with binomial probabilities

Exercise 6: Survey response rates

Is it reasonable to expect that a survey method with a 20% response rate will produce over $2,500$ responses from a sample of $20,000$ households?

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$$\mu = np = (20000)(.2) = 4000$$

$$\sigma^2 = np(1-p) = (20000)(.2)(.8) = 3200 \quad \sigma = \sqrt{3200} \approx 56$$

The distribution approaches $\mathcal{N}(\mu = 4000, \sigma = 56)$, which makes $P(x = 2500)$ an extremely likely outcome.

Exercise with sample size

Determining a margin of error

$$Z \cdot \frac{\sigma}{\sqrt{N}} \leq ME$$

Example: average number of sexual partners

How many people should we sample from a population where the number of sexual partners has an unknown mean and a standard deviation of 3 if we want a margin of error around 1 partner at 95% confidence?

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$$1.96 \cdot \frac{\sigma}{\sqrt{N}} \leq 1 \quad 1.96 \cdot \frac{3}{1} \leq \sqrt{N} \quad (1.96 \cdot 3)^2 \leq N \quad N \geq 35$$